Phys 410
Fall 2015
Lecture #7 Summary
22 September, 2015

We considered two-particle conservative-force interactions and found that the potential function $U(\vec{r}_1 - \vec{r}_2)$ does double duty by representing two forces: $\vec{F}_{12}$ and $\vec{F}_{21}$. In other words, one can recover the forces of interaction between the two particles as $\vec{F}_{12} = -\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)$ and $\vec{F}_{21} = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)$, where the subscripts on the gradients denote derivatives with respect to the coordinates of one of the particles. In addition, this single potential energy function properly keeps track of the total potential energy of the two particles interacting by means of the associated conservative forces (hence it does triple-duty). The total mechanical energy of the two-particle system is given by $E = T_1 + T_2 + U(\vec{r}_1 - \vec{r}_2)$, and this quantity is conserved.

This treatment can be generalized to an arbitrary number of particles interacting with each other by means of conservative forces, and under the influence of external conservative forces. The potential energy due to conservative-force interactions among an N-particle system is

$$U^{int} = \sum_{\alpha=1}^{\alpha=N} \sum_{\beta>\alpha}^{\beta=N} U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta).$$

The most likely setting for a large number of particles interacting with each other under ONLY conservative forces is the "rigid body." This is defined as a set of particles whose relative positions never change as the system evolves in time – no matter how fast it rotates or collides with other objects. This rigid body approximation allows one to ignore non-conservative internal forces that typically require that the particles move through some finite displacement relative to each other (e.g internal friction, turbulence, inelastic deformations, etc.)

We began to discuss the physics of oscillations. Any system with a minimum in the potential energy landscape $U(x)$ can have small harmonic oscillations around the minimum in the potential. Therefore, Hooke's law is a useful approximation for a dynamical system around a stable equilibrium. The Hooke's law spring constant is just the local curvature of the potential at the minimum ($x = x_0$): $k = d^2U/dx^2\Big|_{x_0}$. A one-dimensional harmonic oscillator of mass m obeys the equation $m\ddot{x} = -kx$, where x is the displacement from equilibrium. Dividing through by m and defining the natural oscillation frequency $\omega_0 = \sqrt{k/m}$, the equation becomes $\ddot{x} = -\omega_0^2 x$. This equation can be solved in numerous ways, and the solutions can be written in several canonical forms, including:

1) $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$
2) $x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$

3) $x(t) = A\cos(\omega t - \delta)$

4) $x(t) = \mathrm{Re}[Ae^{i(\omega t-\delta)}]$

All of these forms can be related to each other, as you will prove in homework. Note that each form has two unknown parameters, and they can be determined by two initial conditions, since this is the solution to a second order differential equation.

We also considered the energy in simple harmonic motion. The total mechanical energy is $E = T + U = (m/2)\dot{x}^2 + (k/2)x^2$. Using form 3 above (for example), this can be written as $E = (k/2)A^2$, which is constant. The kinetic and potential energies are both varying with time as $\sin^2(\omega t - \delta)$ and $\cos^2(\omega t - \delta)$, respectively. They both oscillate between 0 and E twice per period of oscillation, and are exactly 180° out of phase. The shuttling of energy back and forth between two different forms (in this case potential and kinetic) is a hallmark of simple harmonic oscillation, and resonance.

We considered un-driven damped oscillations produced by a damping (or drag) force that is linear in velocity $m\ddot{x} + b\dot{x} + kx = 0$. This mechanical oscillator is a direct analog of the electrical oscillator made up of an inductor (L), resistor (R) and capacitor (C) in series. The charge on the capacitor plate $q(t)$ obeys the same equation: $L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$. The analogy is strong, as shown in the following table.

Mechanical Oscillator	**Electrical Oscillator**
Position *x*	Charge on capacitor plate *q*
Mass *m*	Inductance *L*
Damping coefficient *b*	Resistance *R*
Spring constant *k*	Inverse Capacitance 1/*C*
Natural frequency $\omega_0 = [k/m]^{1/2}$	Natural frequency $\omega_0 = 1/[LC]^{1/2}$

Divide the mechanical equation through by mass *m* and define two important rates: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where $2\beta \equiv b/m$, and $\omega_0^2 \equiv k/m$. We think of ω_0 as the natural oscillation frequency of the un-damped harmonic oscillator. We can think of β as the relaxation rate of the damped harmonic oscillator. We tried a solution of the form $x(t) = e^{rt}$, and found an auxiliary equation with solution $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. The general solution is $x(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\ t}\right]$. The form of the solution depends critically on the relative size of the two rates β and ω_0.

1) Un-damped oscillator $\beta = 0$. The radical becomes $\sqrt{-\omega_0^2} = i\sqrt{\omega_0^2} = i\omega_0$, and the solution reverts to our previous results $x(t) = C_1 e^{i\omega_0\ t} + C_2 e^{-i\omega_0\ t}$.